

Estimation of latent variable models for ordinal data via fully exponential Laplace approximation

Silvia Bianconcini and Silvia Cagnone
Department of Statistics, University of Bologna.

Introduction

Latent variable models represent a useful tool in the social sciences where the analyzed constructs cannot be directly observed and, hence, they are not measurable. However, a set of indicators related to each unobserved variable can be measured. They are often coded into a number of ordered categories, so that latent variable models with ordinal variables have to be used. These models can be defined within the Generalized Linear Latent Variable Model (GLLVM) framework (Bartholomew & Knott, 1999; Moustaki & Knott, 2001), according to which the entire set of the responses given by an individual to a certain number of items, called response pattern, is expressed as a function of one or more latent variables through a monotone differentiable link function. The estimation of the model parameters can be obtained by means of a full information maximum likelihood method via the EM algorithm, that guarantees quite accurate estimates (Moustaki, 2000, 2003).

The presence of the latent variables causes problems related to the integration of the likelihood function, since analytical solutions do not exist. In order to overcome this drawback, numerical approximations are usually applied. One of the most often used technique is the classical Gauss-Hermite (GH) quadrature (Bock & Aitkin, 1981), that provides quite good parameter estimates when many quadrature points are considered per each latent variable. However, it becomes computational unfeasible as the number of latent variables increases. This represents a serious limitation for a large number of applications where several observed and latent variables are required (Cagnone et al., 2009).

As alternative solution to GH, the Adaptive Gauss Hermite (AGH) quadrature has been discussed for different models with random effects and/or latent variables (Pinhero & Bates, 1995; Rabe-Hesketh et al., 2005; Schilling & Bock, 2005). In all these studies, AGH is shown to perform better than GH, also when few quadrature points are used. Indeed, it consists of adjusting the GH nodes with the first and second moments of the posterior density of the latent factors given the manifest variables. This allows a better approximation of the function to be integrated. Nevertheless, the AGH is very computational intensive, particularly in latent variable models for ordinal data (Cagnone & Monari, 2011).

An approximation technique that is not affected by the presence of high dimensional integrals is the Laplace method (De Bruijn, 1981; Barndorff-Nielsen & Cox, 1989), that can be viewed as a particular case of the AGH when just one abscissa is used (Liu & Pierce, 1994). Given its reduced dimensionality, the Laplace method is one of the fastest technique, since the computational burden depends only on the calculation of the mode of the integrand

(Tierney & Kadane, 1986; Raudenbush et al., 2000; Huber et al., 2004; Pinhero & Chao, 2006). However, the Laplace approximation has an error of order $O(p^{-1})$, that depends only on the number of items p , hence it is not directly controllable. Moreover, Joe (2008) has investigated its performance for a variety of discrete response mixed models, and he has found that it becomes less adequate as the degree of discreteness increases.

When either the EM algorithm or a direct maximization of the observed data log-likelihood is used for model estimation, an extended version of the Laplace method, called Fully exponential Laplace Approximation (FLA), can be applied. It has been introduced and developed by Tierney et al. (1989) in the Bayesian context for approximating posterior distributions. Recently, it has been extended by Rizopoulos et al. (2009) to a variety of models for longitudinal continuous measurements and time-to-event data estimated via the EM algorithm. The main idea proposed by these authors is to apply the FLA to the expected score function of the model parameters with respect to the posterior distribution of the latent variables. With the FLA, a better approximation of the multidimensional integrals is achieved, being the approximation error of order $O(p^{-2})$. Moreover, the computational complexity of this approach is similar to the classical Laplace method since it depends only on the numerical optimization required to compute the mode of the integrand.

In this paper, we extend the FLA for the general class of latent variable models for ordinal data within the GLLVM context. In Section 2, the models for ordinal data are introduced, whereas in Section 3 the estimation problem is discussed, with particular attention to the fully exponential Laplace approximation. In Section 4, a simulation study is performed in order to compare the finite sample and asymptotic properties of the AGH and FLA under different conditions. Finally, Section 5 gives the conclusions.

Model specification

Let \mathbf{y} be a vector of p ordinal observed variables each of them with c_i categories, and \mathbf{z} be a vector of q latent variables. The c_i ($i = 1, \dots, p$) ordered categories of the variables y_i have associated the probabilities $\pi_{i,1}(\mathbf{z}), \pi_{i,2}(\mathbf{z}), \dots, \pi_{i,c_i}(\mathbf{z})$, which are functions of the vector of the latent variables \mathbf{z} .

Following the general scheme of the GLLVM framework, the probability associated to \mathbf{y} is given by

$$f(\mathbf{y}) = \int_{R^q} g(\mathbf{y}|\mathbf{z})h(\mathbf{z})d\mathbf{z} \quad (1)$$

where $h(\mathbf{z})$ is assumed to be a multivariate standard normal distribution. $g(\mathbf{y}|\mathbf{z})$ is the conditional probability of the observed variables given \mathbf{z} . It is assumed to follow a multinomial distribution

$$g(\mathbf{y} | \mathbf{z}) = \prod_{i=1}^p g(y_i | \mathbf{z}) \quad (2)$$

where

$$g(y_i | \mathbf{z}) = \prod_{s=1}^{c_i-1} \left(\frac{\gamma_{i,s}(\mathbf{z})}{\gamma_{i,s+1}(\mathbf{z}) - \gamma_{i,s}(\mathbf{z})} \right)^{y_{i,s}^*} \left(\frac{\gamma_{i,s+1}(\mathbf{z}) - \gamma_{i,s}(\mathbf{z})}{\gamma_{i,s+1}(\mathbf{z})} \right)^{y_{i,s+1}^* - y_{i,s}^*}. \quad (3)$$

Expression (2) is obtained by assuming the conditional independence of the observed variables given the latent variables. In expression (3), $\gamma_{i,s}(\mathbf{z}) = \pi_{i,1}(\mathbf{z}) + \pi_{i,2}(\mathbf{z}) + \dots + \pi_{i,s}(\mathbf{z})$

is the probability of a response in category s or lower on the variable i , and it is function of \mathbf{z} . For simplicity, from now on we consider $\gamma_{i,s} = \gamma_{i,s}(\mathbf{z})$. $y_{i,s}^*$ is equal to 1 if the response y_i is in the category s or lower, and 0 otherwise.

As in the classical generalized linear model, the systematic component is defined as

$$\eta_{i,s} = \tau_{i,s} - \sum_{j=1}^q \alpha_{ij} z_j, \quad s = 1, \dots, c_i - 1, i = 1, \dots, p \quad (4)$$

where $\eta_{i,s}$ is the linear predictor, and $\tau_{i,s}$ and α_{ij} can be interpreted as thresholds and factor loadings of the model. For the thresholds, the inequality $-\infty = \tau_{i,0} \leq \tau_{i,1} \leq \tau_{i,2} \leq \dots \leq \tau_{i,c_i} = +\infty$ holds. Each factor loading α_{ij} measures the effect of the correspondent latent variable z_j on some function of the cumulative probability $\gamma_{i,s}$.

The relation between the systematic component and the conditional means of the random component distributions is given by $\eta_{i,s} = \nu_{i,s}(\gamma_{i,s})$, where $\nu_{i,s}$ is the link function and can be any monotonic differentiable function. Here, we refer to the logit link function, so that eq. (4) is known as proportional odds model. However, other link functions can be chosen.

Model estimation

Model estimation is achieved by using the maximum likelihood through the EM algorithm, since the latent variables are unknown. At this regard, we apply a full information maximum likelihood method by which all the parameters of the model are estimated simultaneously.

For a random sample of size n , from equation (1), the observed data log-likelihood is defined as

$$L = \sum_{l=1}^n \log f(\mathbf{y}_l) = \sum_{l=1}^n \log \int_{R^q} g(\mathbf{y}_l | \mathbf{z}_l) h(\mathbf{z}_l) d\mathbf{z}_l. \quad (5)$$

The EM algorithm consists of an Expectation step (E-step), in which the expected score function $E(S(\mathbf{a}_i))$ of the model parameters $\mathbf{a}_i' = (\tau_{i,1}, \dots, \tau_{i,c_i-1}, \alpha_{i1}, \dots, \alpha_{iq})$, $i = 1, \dots, p$, is computed. The expectation is with respect to the posterior distribution $h(\mathbf{z}_l | \mathbf{y}_l)$ of \mathbf{z} given the observations for each individual. In the Maximization step (M-step), updated parameter estimates are obtained by equating to 0 the expected score functions.

Louis (1982) proved that maximizing the observed data score vector $\partial L / \partial \mathbf{a}_i$ is equivalent to maximize the expected score function $E(S(\mathbf{a}_i))$ with respect to $h(\mathbf{z}_l | \mathbf{y}_l)$, so that

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{a}_i} = E(S(\mathbf{a}_i)) &= \sum_{l=1}^n \frac{\int S(\mathbf{a}_i) g(\mathbf{y}_l | \mathbf{z}_l) h(\mathbf{z}_l) d\mathbf{z}_l}{\int g(\mathbf{y}_l | \mathbf{z}_l) h(\mathbf{z}_l) d\mathbf{z}_l} = \\ &= \sum_{l=1}^n \int \frac{\partial \log g(\mathbf{y}_l | \mathbf{z}_l)}{\partial \mathbf{a}_i} h(\mathbf{z}_l | \mathbf{y}_l) d\mathbf{z}_l \quad i = 1, \dots, p \end{aligned} \quad (6)$$

where

$$S(\mathbf{a}_i) = \frac{\partial \log g(\mathbf{y}_l | \mathbf{z}_l)}{\partial \mathbf{a}_i} = \sum_{s=1}^{c_i-1} \left[y_{i,s,l}^* \frac{\partial \theta_{i,s,l}(\mathbf{z}_l)}{\partial \mathbf{a}_i} - y_{i,s+1,l}^* \frac{\partial b(\theta_{i,s,l}(\mathbf{z}_l))}{\partial \mathbf{a}_i} \right]$$

and

$$\theta_{i,s,l}(\mathbf{z}_l) = \log \frac{\gamma_{i,s,l}}{(\gamma_{i,s+1,l} - \gamma_{i,s,l})} \quad s = 1, \dots, c_i - 1, i = 1, \dots, p. \quad (7)$$

$$b(\theta_{i,s,l}(\mathbf{z}_l)) = \log \frac{\gamma_{i,s+1,l}}{(\gamma_{i,s+1,l} - \gamma_{i,s,l})} \quad s = 1, \dots, c_i - 1, i = 1, \dots, p. \quad (8)$$

The expressions of the derivatives reported in the last equality of (7) with respect to thresholds and loadings can be found in Moustaki (2000, 2003).

From eq. (6), it can be noticed that the computation of the expected score functions involves a multidimensional integral that cannot be solved analytically, hence numerical approximations are required. In particular, in the following, we propose the use of an extended version of the classical Laplace approximation, that is the fully exponential Laplace method.

Fully exponential Laplace approximation method

The FLA method has been proposed for the first time by Tierney et al. (1989) in order to approximate posterior distributions in the Bayesian context. It represents an extension of the classical Laplace approximation that, as known, is based on the second order Taylor expansion of the logarithm of the integrand, with the latent variables evaluated at the mode (see, among the others, Tierney & Kadane (1986)).

The Laplace method has the advantage of dealing with integrals of any dimensionality without introducing computational problems but, for the general class of latent variable models discussed in this paper, it produces an approximation error of order $O(p^{-1})$, that can be reduced only increasing the number of observed variables. The FLA leads to an improvement of the approximation error maintaining the same computational complexity as the classical Laplace method. The extension of FLA to joint models for continuous longitudinal measurements and time-to-event data has been proposed by Rizopoulos et al. (2009). It requires the computation of the following quantities

$$E(A(\mathbf{z}_l)) = \int A(\mathbf{z}_l)h(\mathbf{z}_l | \mathbf{y}_l)d\mathbf{z}_l = \frac{\int A(\mathbf{z}_l)g(\mathbf{y}_l | \mathbf{z}_l)h(\mathbf{z}_l)d\mathbf{z}_l}{\int g(\mathbf{y}_l | \mathbf{z}_l)h(\mathbf{z}_l)d\mathbf{z}_l} \quad (9)$$

that differ from (6) since $A(\cdot)$ are the components of the score functions $S(\mathbf{a}_i)$ that depend on the latent variables.

The main idea of FLA is to approximate both the numerator and the denominator in eq. (9) with the classical Laplace method. Tierney & Kadane (1986) proved that the error terms of order $O(p^{-1})$ in the numerator and the denominator cancel out, leading to a smaller error term of order $O(p^{-2})$.

To extend the FLA to the proportional odds model discussed in this paper, we have to take into account for the derivatives (7) with respect to the thresholds and the loadings, that are characterized by different $A(\mathbf{z}_l)$ components. In more detail, from the derivatives of the logarithm of $g(y_{i,l} | \mathbf{z}_l)$ with respect to the thresholds we get

$$A_{1,i,s}(\mathbf{z}_l) = -\frac{\partial \theta_{i,s-1,l}(\mathbf{z}_l)}{\partial \tau_{i,s}} = \begin{cases} (1 - \gamma_{i,s,l}), & \text{if } s = 1; \\ (1 - \gamma_{i,s,l}) \frac{\gamma_{i,s,l}}{(\gamma_{i,s,l} - \gamma_{i,s-1,l})}, & \text{if } s = 2, \dots, c_i - 1. \end{cases}$$

$$A_{2,i,s}(\mathbf{z}_l) = \frac{\partial b(\theta_{i,s,l}(\mathbf{z}_l))}{\partial \tau_{i,s}} = (1 - \gamma_{i,s,l}) \frac{\gamma_{i,s,l}}{(\gamma_{i,s+1,l} - \gamma_{i,s,l})} \quad s = 1, \dots, c_i - 1.$$

From the derivatives of the logarithm of $g(y_{i,l} \mid \mathbf{z}_l)$ with respect to the loadings $\boldsymbol{\alpha}_i = (\alpha_{i1}, \dots, \alpha_{iq})$, we get

$$A_{3,i,s}(\mathbf{z}_l) = -\frac{\partial \log g(y_{i,s,l} \mid \mathbf{z}_l)}{\partial \alpha_i} = \begin{cases} (1 - \gamma_{i,s,l})\mathbf{z}_l, & \text{if } s = 1; \\ (1 - \gamma_{i,s,l} - \gamma_{i,s-1,l})\mathbf{z}_l, & \text{if } s = 2, \dots, c_i. \end{cases}$$

The FLA approximation can be applied only to strictly positive functions $A(\cdot)$. In our case, this condition is not necessarily guaranteed since the $A(\cdot)$ are components of the score functions, not constrained to be positive. To overcome this problem, the method of the moment generating function can be used. According to this approach, since the quantity $\exp\{t'A(\mathbf{z}_l)\}$ is always positive, the FLA approximation can be applied to the moment generating function $M(t) = E[\exp\{t'A(\mathbf{z}_l)\}]$, with latent variables \mathbf{z}_l evaluated at the mode $\hat{\mathbf{z}}_l = \arg \max_{\mathbf{z}_l} [\log g(\mathbf{y}_l \mid \mathbf{z}_l) + \log h(\mathbf{z}_l) + t'A(\mathbf{z}_l)]$. In doing so, we get the approximate moment generating function $\hat{M}(t)$. Hence, from the corresponding cumulant-generating function $\log \hat{M}(t)$, we obtain the approximate expected values $\hat{E}(A(\mathbf{z}_l))$. These latter are the quantities of interest, and they are given by

$$\hat{E}(A(\mathbf{z}_l)) = \frac{\partial}{\partial t} \log \hat{M}(t)|_{t=0} = \frac{\partial}{\partial t} \log \hat{E}[\exp\{t'A(\mathbf{z}_l)\}]|_{t=0}. \quad (10)$$

Tierney et al. (1989) proved (Theorem 2, pag. 712) that eq. (10) is equivalent to the following expression

$$\begin{aligned} \hat{E}(A(\mathbf{z}_l)) &= A(\hat{\mathbf{z}}_l) + \frac{\partial \log \det(\boldsymbol{\Sigma}_l^{(t)})^{-1/2}}{\partial t} \Big|_{(\mathbf{z}_l=\hat{\mathbf{z}}_l, t=0)} + O(p^{-2}) = \\ &= A(\hat{\mathbf{z}}_l) - \frac{1}{2} \text{tr}(\boldsymbol{\Omega}) \Big|_{(\mathbf{z}_l=\hat{\mathbf{z}}_l, t=0)} + O(p^{-2}) \end{aligned} \quad (11)$$

where

$$\begin{aligned} \boldsymbol{\Sigma}_l^{(t)} &= -\frac{\partial^2 \{\log g(\mathbf{y}_l \mid \mathbf{z}_l) + \log h(\mathbf{z}_l) + t'A(\mathbf{z}_l)\}}{\partial \mathbf{z}_l' \partial \mathbf{z}_l} = \\ &= \sum_{i=1}^p \sum_{s=1}^{c_i-1} \{ \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i' [-y_{i,s,l}^* \gamma_{i,s+1,l} (1 - \gamma_{i,s+1,l}) + y_{i,s+1,l}^* \gamma_{i,s,l} (1 - \gamma_{i,s,l})] \} + \mathbf{I} - \frac{\partial^2 t'A(\mathbf{z}_l)}{\partial \mathbf{z}_l' \partial \mathbf{z}_l} \end{aligned} \quad (12)$$

and

$$\boldsymbol{\Omega} = \boldsymbol{\Sigma}_l^{-1} \{ \partial \boldsymbol{\Sigma}_l^{(t)} / \partial t \}.$$

The expressions of the first derivatives of $\boldsymbol{\Sigma}_l$ with respect to t are reported in the Appendix.

EM algorithm

The steps of the EM algorithm are defined as follows:

1. Choose initial values for the parameters $\mathbf{a}_i' = (\tau_{i,1}, \dots, \tau_{i,c_i-1}, \alpha_{i1}, \dots, \alpha_{iq}) \quad i = 1, \dots, p$.

2. Compute the mode $\hat{\mathbf{z}}_l, l = 1, \dots, n$, by using a Newton Raphson iteration scheme. In more detail, for the (m) -th iteration

$$\hat{\mathbf{z}}_l^m = \hat{\mathbf{z}}_l^{(m-1)} - [(\boldsymbol{\Sigma}_l^{(m-1)})^{-1} S(\mathbf{z}_l^{(m-1)})]_{(\mathbf{z}_l = \hat{\mathbf{z}}_l^{(m-1)}, t=0)}$$

where $\boldsymbol{\Sigma}_l$ is the Hessian matrix defined in expression (12) and $S(\mathbf{z}_l)$ is defined as follows

$$S(\mathbf{z}_l) = - \frac{\partial \{\log g(\mathbf{y}_l | \mathbf{z}_l) + \log h(\mathbf{z}_l) + t' A(\mathbf{z}_l)\}}{\partial \mathbf{z}_l} = \quad (13)$$

$$= \sum_{i=1}^p \sum_{s=1}^{c_i-1} \{ \boldsymbol{\alpha}'_i [y_{i,s,l}^* \gamma_{i,s+1,l} - y_{i,s+1,l}^* \gamma_{i,s,l}] \} + \mathbf{z}_l - \frac{\partial t' A(\mathbf{z}_l)}{\partial \mathbf{z}_l}. \quad (14)$$

3. E-step. Compute the FLA expected values $\hat{E}(A_{1,i,s}(\mathbf{z}_l))$, $\hat{E}(A_{3,i,s}(\mathbf{z}_l))$, for $s = 1, \dots, c_i - 1, i = 1, \dots, p$, and $\hat{E}(A_{2,i,s-1}(\mathbf{z}_l))$ for $s = 2, \dots, c_i, i = 1, \dots, p$, and the approximate expected score function $\hat{E}(S(\mathbf{a}_i))$, where $i = 1, \dots, p$.
4. M-step. Obtain improved estimates for the model parameters $\mathbf{a}'_i = (\tau_{i,1}, \dots, \tau_{i,c_i-1}, \alpha_{i1}, \dots, \alpha_{iq})$, $i = 1, \dots, p$. For all of them, a Newton Raphson iterative scheme is used in order to solve the corresponding nonlinear maximum likelihood equations.
5. Repeat steps 2-3-4 until convergence is attained.

Simulation study

The properties of the FLA method for the proportional odds model can be evaluated by performing a simulation study in which several conditions are taken into account. The results will be compared with those obtained using the AGH quadrature. In recent years, the latter has been widely applied in latent variable models, since it allows to obtain estimates that are as accurate as those derived by the GH technique, but using a small number of quadrature points. It essentially consists of scaling and translating the classical Gaussian quadrature locations to place them under the peak of the integrand, and two different procedures have been adopted in the literature. According to the first one, the mode of the integrand and the inverse of the information matrix of the integrand evaluated at the mode are computed (Liu & Pierce, 1994; Pinheiro & Bates, 1995; Schilling & Bock, 2005). The advantage of this approach lies in the fact that the quadrature points are not involved in these computations. However, this method is computationally demanding since it requires numerical optimization routines and the computation of second derivatives. Moreover, when parameter estimates are obtained by using iterative algorithms, like in our case, the first and second order moments have to be computed at each step, hence the algorithm becomes very slow.

An alternative procedure consists of computing the posterior means and covariance matrices at each step of the algorithm (Rabe-Hesketh et al., 2005). Although this method requires the use of quadrature points themselves, the posterior moments should better describe the integrand in those cases in which its tails are heavier than the normal density. In the following, we show how both these techniques work in latent variable models for ordinal data, and we compare their performances with FLA.

The softwares used for the analyses are Fortran 95 and R. The codes are available from the authors upon request.

Table 1: Mean, bias and MSE of the parameter estimates for FLA, AGH_{me}, and AGH_{mo} in the generated data.

% valid samples	AGH						FLA		
	mean			mode					
	87			84			76		
True	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
$\alpha_{11} = 1.03$	1.52	0.49	0.96	1.52	0.49	0.68	1.05	0.02	0.03
$\alpha_{21} = 1.44$	1.48	0.04	0.53	1.20	-0.24	0.38	1.37	-0.07	0.14
$\alpha_{31} = 2.11$	2.19	0.08	0.82	1.88	-0.23	0.44	2.08	-0.03	0.31
$\alpha_{41} = 1.80$	1.81	0.01	0.63	1.65	-0.15	0.44	1.44	-0.36	0.21
$\alpha_{51} = 1.53$	1.53	0.00	0.42	1.36	-0.17	0.27	1.54	0.01	0.14
$\alpha_{12} = 0.00$	-	-	-	-	-	-	-	-	-
$\alpha_{22} = 2.42$	2.04	-0.38	0.60	2.10	-0.33	0.41	2.04	-0.38	0.35
$\alpha_{32} = 1.52$	1.62	0.10	0.56	1.82	0.30	0.45	2.02	0.50	0.51
$\alpha_{42} = 0.75$	0.78	0.03	0.44	0.96	0.21	0.36	1.11	0.36	0.18
$\alpha_{52} = 1.34$	1.50	0.16	0.38	1.63	0.29	0.41	1.76	0.42	0.35

Finite sample properties of the estimators

To investigate empirically the finite sample performance of the FLA and AGH, based on both the posterior mean (AGH_{me}) and mode (AGH_{mo}), we generated data from a population that consists of five variables and satisfies a two factor model. The number of categories is the same for each observed variable, and equal to 4. 100 random samples were considered with $n = 200$ subjects. We chose 5 quadrature points per each latent variable for both the adaptive approximations. We also considered 7 quadrature points, but there was a little difference with 5 nodes, suggesting that the latter provides sufficient accuracy for this example.

The population parameters were chosen in such a way that the thresholds range from -3 to 3. The factor loadings are the following: $\alpha_1 = (1.03, 1.44, 2.11, 1.8, 1.53)$ and $\alpha_2 = (0, 2.42, 1.52, 0.75, 1.34)$ with not null values generated from a log-normal distribution, and one loading fixed to 0 to get a unique solution.

Table 1 reports the mean, bias, and Mean Square Error (MSE) of the parameter estimates obtained by applying all the techniques. The results show that the percentage of valid samples is quite high for all the procedures, ranging from 76% to 87%. The FLA presents much better MSE values than those achieved by AGH_{me} and AGH_{mo}, mainly due to a smaller variability of the estimates. Comparing the adaptive techniques, AGH_{me} estimates are less biased than those determined by AGH_{mo}, and present an opposite sign of the bias for α_1 . On the other hand, the latter behaves better in terms of MSE values.

The different performance of the two adaptive techniques can be due to the fact that the individual posterior densities to be approximated are not always symmetric. In latent variable models for ordinal data, Chang (1996) proved that the posterior densities asymptotically follow a multivariate normal distribution. However, for a small number of observed variables, the integrand could be skewed, and the numerical procedures could provide quite different results.

To analyze the shape of the individual posterior densities in the generated population, we computed measures of multivariate skewness $\beta_{1,q}$ and kurtosis $\beta_{2,q}$ proposed by Mardia (1970). In the case of two latent variables, they are given by

$$\beta_{1,2}(l) = \mu_{30}^2 + \mu_{03}^2 + 3\mu_{12}^2 + 3\mu_{21}^2 \quad l = 1, \dots, n$$

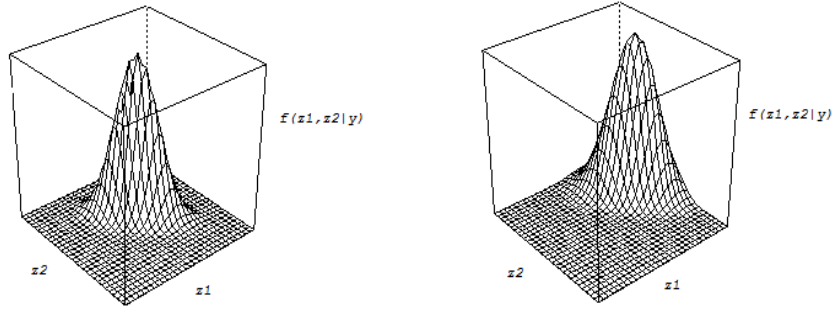
and

$$\beta_{2,2}(l) = \mu_{04} + \mu_{40} + 2\mu_{22} \quad l = 1, \dots, n$$

with $\mu_{ij} = E(z_{1l}^i z_{2l}^j)$, whereas z_{1l} and z_{2l} are the latent factors standardized with respect to the posterior densities. Mardia (1970) also derived the asymptotic distributions of both $\beta_{1,q}$ and $\beta_{2,q}$, and the corresponding statistical tests to evaluate the null hypotheses $H_0 : \beta_{1,q} = 0$ and $H_0 : \beta_{2,q} = q(q+2)$, being $q(q+2)$ the kurtosis in q -variate normal densities.

By computing these measures for the individual posterior densities generated in this simulation study, we observed that about 35% of these functions have a significant skewness, and a kurtosis always not significantly different from 8. In particular, $\beta_{1,2}$ is on average equal to 0.044 and it ranges from 0.000 to the significant value 0.168. The presence of individual posterior densities having different shapes could justify the different behavior of AGHs and FLA. In Figure 1, we show two different functions obtained from our generated data.

Figure 1. Individual posterior densities with different shapes (on the left side $\beta_{1,2}=0.000$, and on the right side $\beta_{1,2}=0.168$) in the generated population of 200 subjects.



In order to better analyze the finite sample properties of FLA and AGHs, we also generated data from two hypothetical extreme scenarios: one in which all the posterior densities are symmetric, and another one in which a high percentage (more than 60%) of the densities are skewed. As before, we consider five observed variables, each with 4 categories, satisfying a two factor model. The results for both the populations are shown in Table 2.

In the first scenario, the thresholds for each item are equal to -2 for the first category, 0 for the second, and 2 for the third one, whereas the loadings are all fixed to 0.5 except one set equal to zero. In this population, all the individual posterior densities are symmetric, with $\beta_{1,2}$ on average equal to 0.005, and $\beta_{2,2}$ always not significantly different from 8. As in the previous simulation study, we generated 100 random samples with 200 subjects.

For all the samples the algorithm achieves the convergence for FLA and AGH_{me} , and in the 96% of the cases for AGH_{mo} . The FLA improves a lot with respect to the previous case, with a reduction of almost one digit in the MSE values, mainly due to smaller bias values for α_2 . On the other hand, both the AGH techniques provide better results in terms of bias and MSE, even if they still perform worse than FLA. We can also notice that the results

Table 2: Mean, bias and MSE for FLA, AGH_{me}, and AGH_{mo} for different scenarios in finite samples.

	AGH						FLA		
	mean			mode					
% valid samples	100			96			100		
True	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
$\alpha_{11} = 0.5$	0.78	0.28	0.59	0.79	0.29	0.62	0.59	0.09	0.02
$\alpha_{21} = 0.5$	0.46	-0.04	0.39	0.50	-0.00	0.46	0.61	0.11	0.02
$\alpha_{31} = 0.5$	0.43	-0.07	0.50	0.41	-0.09	0.48	0.60	0.10	0.02
$\alpha_{41} = 0.5$	0.52	0.02	0.78	0.54	0.04	0.77	0.60	0.10	0.02
$\alpha_{51} = 0.5$	0.61	0.11	0.67	0.60	0.10	0.64	0.61	0.11	0.02
$\alpha_{12} = 0.0$	-	-	-	-	-	-	-	-	-
$\alpha_{22} = 0.5$	0.81	0.31	0.80	0.83	0.33	0.82	0.60	0.10	0.01
$\alpha_{32} = 0.5$	0.64	0.14	0.79	0.58	0.08	0.66	0.59	0.09	0.01
$\alpha_{42} = 0.5$	0.80	0.30	0.77	0.77	0.27	0.85	0.59	0.09	0.01
$\alpha_{52} = 0.5$	0.72	0.22	0.61	0.74	0.24	0.65	0.59	0.09	0.01

	AGH						FLA		
	mean			mode					
% valid samples	83			27			35		
True	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
$\alpha_{11} = 2.5$	2.49	-0.01	0.31	2.35	-0.16	0.18	1.82	-0.68	0.60
$\alpha_{21} = 2.5$	2.74	0.24	0.37	2.56	0.06	0.15	2.42	-0.08	0.19
$\alpha_{31} = 2.5$	2.79	0.29	0.58	2.47	-0.03	0.16	2.36	-0.14	0.16
$\alpha_{41} = 2.5$	2.63	0.13	0.24	2.57	0.07	0.12	2.41	-0.09	0.08
$\alpha_{51} = 2.5$	2.75	0.25	0.41	2.61	0.11	0.14	2.37	-0.13	0.15
$\alpha_{12} = 0.0$	-	-	-	-	-	-	-	-	-
$\alpha_{22} = 1.0$	1.04	0.04	0.44	0.90	-0.10	0.35	1.86	0.86	0.81
$\alpha_{32} = 1.0$	1.26	0.26	0.52	0.94	-0.07	0.34	1.82	0.82	0.76
$\alpha_{42} = 1.0$	1.07	0.07	0.50	1.09	0.09	0.65	1.90	0.90	0.87
$\alpha_{52} = 1.0$	1.23	0.23	0.65	0.94	-0.06	0.61	1.87	0.87	0.80

provided by the two adaptive procedures are almost the same, with an equal sign of the bias for all the estimates, and slight discrepancies due to the different computational techniques involved. Indeed, as discussed by Rabe-Hesketh et al. (2005), the two procedures should provide similar results when the posterior densities are symmetric.

In the second scenario, the thresholds for each item are equal to -1 for the first category, 0 for the second, and 1 for the third one, whereas the loadings are fixed equal to $\alpha_1 = (2.5, 2.5, 2.5, 2.5, 2.5)$ and $\alpha_2 = (0, 1, 1, 1, 1)$. In this case, the 65% of the posterior densities are skewed. $\beta_{1,2}$ ranges from 0.000 to 0.239, being the latter significantly different from zero, and it is on average equal to 0.127. On the other hand, there is not significant kurtosis for all the subjects. The main consequence of this high percentage of skew densities is that, for both FLA and AGH_{mo}, a very small number of samples (27% for the former, 35% for the latter) converge properly. Hence, even if the results are similar to the ones obtained in the first simulation, they are not reliable. On the other hand, AGH_{me} seems to be not affected by the different shapes of the posterior densities. It results more stable in terms of mean, bias, and MSE of the estimates as well as in terms of percentage of valid samples, that also in this case is 83%.

From these results, we can argue that FLA will be superior than AGH when the majority of the posterior densities is symmetric. In these cases the former provides better MSE values for the estimates than the latter, mainly due to a reduced variability in the estimates. Moreover, the bias introduced in the estimates using FLA is quite comparable with the one

Table 3: Mean, bias and MSE for FLA, AGH_{me}, and AGH_{mo} for generated data with $n = 1000$.

	AGH						FLA		
	mean			mode					
% valid samples	97			92			99		
True	Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
$\alpha_{11} = 1.03$	1.11	0.08	0.11	1.13	0.10	0.09	1.04	0.01	0.01
$\alpha_{21} = 1.44$	1.43	-0.01	0.10	1.39	-0.05	0.10	1.39	-0.05	0.04
$\alpha_{31} = 2.11$	2.12	0.01	0.09	2.09	-0.02	0.09	2.15	0.04	0.07
$\alpha_{41} = 1.80$	1.85	0.05	0.12	1.83	0.03	0.13	1.50	0.05	0.04
$\alpha_{51} = 1.53$	1.55	0.02	0.06	1.52	-0.01	0.06	1.59	0.05	0.04
$\alpha_{12} = 0.00$	-	-	-	-	-	-	-	-	-
$\alpha_{22} = 2.42$	2.17	-0.25	0.20	2.23	-0.19	0.19	2.16	-0.26	0.15
$\alpha_{32} = 1.52$	1.56	0.04	0.11	1.61	0.09	0.12	2.04	0.52	0.32
$\alpha_{42} = 0.75$	0.74	-0.01	0.09	0.78	0.03	0.09	1.05	0.30	0.10
$\alpha_{52} = 1.34$	1.39	0.05	0.07	1.42	0.08	0.07	1.73	0.39	0.20

in the AGH estimates. On the other hand, we have also shown that the AGH_{me} provides more stable results, that are not affected by the shape of the integrand. Its use is then suggested in populations characterized by a high percentage of skew distributions.

Asymptotic properties of estimators

The asymptotic properties of the Laplace maximum likelihood estimators $\hat{\theta}$ have been derived and discussed by Rizopoulos et al. (2009). Under suitable regularity conditions, these authors showed that

$$\hat{\theta} - \theta_0 = O_p \left[\max \left\{ n^{-1/2}, p^{-2} \right\} \right],$$

where θ_0 denotes the true parameter value. $\hat{\theta}$ will be consistent as long as both n and p grow to ∞ . FLA is superior than standard Laplace method, the latter producing estimators with an approximation error of order $O_p \left[\max \left\{ n^{-1/2}, p^{-1} \right\} \right]$. On the other hand, following Liu & Pierce (1994) and Tierney et al. (1989), it can be shown that FLA shares the same approximation error of the AGH with 5 quadrature points.

To assess the asymptotic accuracy of the FLA estimators, we generated 100 random samples with 1000 subjects from the population described in the previous section. We also applied both the adaptive techniques, and the results are shown in Table 3.

The percentage of valid samples is high for all the techniques, ranging from 92% to 99%. FLA has a good performance as before with small MSE and bias values. On the other hand, both AGHs have an analogous behavior: the MSE values are drastically reduced with respect to the finite sample situation, and the bias is small for all the parameters.

The three techniques present a very similar asymptotic behavior. Moreover, it is worth noting that FLA performs better than the classical Laplace approximation. Indeed, the asymptotic bias of the latter is higher than the one corresponding to AGH (Joe, 2008), whereas the bias in the AGH and FLA estimates is quite comparable, with a slight better performance of the former for the second factor loadings (Table 3).

Discussion

This paper is concerned with the adequacy of several approximations of the likelihood function in latent variable models for ordinal data. In particular, we proposed an

extended version of the Laplace method for approximating integrals, known as fully exponential Laplace approximation. Classical Laplace methods are known to work poorly in presence of discrete response variables (Joe, 2008), but we have shown how the FLA is generally appropriate in models for ordinal data in both finite and large samples. The comparison with the adaptive Gauss Hermite quadrature techniques has highlighted that in finite samples the FLA provides better results in terms of MSE values when the majority of the posterior densities is symmetric. Indeed, for a small number of observed variables, the symmetry of the individual posterior densities is not always guaranteed, and the percentage of skew distributions tends to vary according to the parameter values. When the majority of the densities are skewed, FLA and AGH_{mo} do not achieve converge in most cases. On the other hand, AGH_{me} is more stable, and it is not affected by the shape of the functions to be approximated.

The main strength of the FLA approach is that it effectively copes with high dimensional latent structures without increasing substantially the computational burden. This is one of the main drawbacks in the application of AGH techniques in latent variable models. Five quadrature points can provide accurate estimates, but the computational effort increases exponentially as the number of latent factors increases. Furthermore, in large samples, the FLA achieves the same approximation of the AGH with five quadrature points, and all the techniques behave similarly.

The main limitation of the FLA approach is that it is not possible to control the magnitude of the approximation error of the integral, as done in AGH by modifying the number of quadrature points. However, as discussed by Rizopoulos et al. (2009), a virtue of the fully exponential Laplace approximation is that it is very general, and it can be used in almost all the general linear latent variable models. Overall, for latent variable models with ordinal data, the FLA is very adequate to approximate the likelihood function, and it should be considered as a valid alternative to adaptive Gaussian quadrature techniques.

Further lines of research will be oriented to compare the performance of FLA with the multidimensional splines. The latter represents a useful alternative to approximate the posterior densities (Thissen & Woods, 2006) and to investigate the main assumptions on the prior distribution of the latent variables that is still an open issue in the GLLVM framework (Knott & Tzamourani, 2007).

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Appendix

In order to apply the fully exponential Laplace approximation, the first derivative of Σ_l with respect to t has to be computed. At this regard, we make use of the following result

$$\left. \frac{\partial \Sigma_l^{(t)}}{\partial t} \right|_{(\mathbf{z}_l = \hat{\mathbf{z}}_l, t=0)} = \frac{\partial \Sigma_l^{(t)}}{\partial \mathbf{z}} \frac{\partial \hat{\mathbf{z}}_l}{\partial t} \Big|_{(t=0)}$$

according to which

$$\begin{aligned} \left. \frac{\partial \Sigma_l^{(t)}}{\partial t} \right|_{(\mathbf{z}_l = \hat{\mathbf{z}}_l, t=0)} &= \sum_{i=1}^p \sum_{s=1}^{c_i-1} \alpha_i \alpha'_i [y_{i,s,l} \gamma_{i,s+1,l} (1 - 3\gamma_{i,s+1,l} + 2\gamma_{i,s+1,l}^2) + \\ &\quad - y_{i,s+1,l} \gamma_{i,s,l} (1 - 3\gamma_{i,s,l} + 2\gamma_{i,s,l}^2)] \times \alpha_i \Sigma_l^{-1} A'(\hat{\mathbf{z}}_l) - A''(\hat{\mathbf{z}}_l), \end{aligned}$$

where $A'(\hat{\mathbf{z}}_l) = \left. \frac{\partial A(\mathbf{z}_l)}{\partial \mathbf{z}_l} \right|_{\mathbf{z}_l = \hat{\mathbf{z}}_l}$ and $A''(\hat{\mathbf{z}}_l) = \left. \frac{\partial^2 A(\mathbf{z}_l)}{\partial \mathbf{z}_l' \partial \mathbf{z}_l} \right|_{\mathbf{z}_l = \hat{\mathbf{z}}_l}$.

For the thresholds, the first-order partial derivatives result

$$A'_{1,i,s}(\mathbf{z}_l) = \alpha_i \gamma_{i,s,l} (1 - \gamma_{i,s,l}), \quad s = 1, \dots, c_i - 1$$

and

$$A'_{2,i,s}(\mathbf{z}_l) = -\alpha_i \gamma_{i,s,l} (1 - \gamma_{i,s,l}) \quad s = 1, \dots, c_i - 1.$$

for $A_{1,i,s}(\mathbf{z}_l)$ and $A_{2,i,s}(\mathbf{z}_l)$, respectively. Furthermore, the corresponding second-order partial derivatives are given by

$$A''_{1,i,s}(\mathbf{z}_l) = -\alpha'_i \alpha_i \gamma_{i,s,l} (1 - 3\gamma_{i,s,l} + 2\gamma_{i,s,l}^2), \quad s = 1, \dots, c_i - 1$$

and

$$A''_{2,i,s}(\mathbf{z}_l) = \alpha'_i \alpha_i \gamma_{i,s,l} (1 - 3\gamma_{i,s,l} + 2\gamma_{i,s,l}^2) \quad s = 1, \dots, c_i - 1.$$

As for the loadings, the elements of the gradient $A_{3,i,s}(\mathbf{z}_l)$ with respect to the latent variables result

$$\begin{aligned} \frac{\partial A_{3,i,s}(\mathbf{z}_{jl})}{\partial z_{jl}} &= \begin{cases} (1 - \gamma_{i,1,l})(1 + \gamma_{i,1,l} \alpha_{ij} z_{jl}), & \text{if } s = 1; \\ (1 - \gamma_{i,s,l} - \gamma_{i,s-1,l}) + \alpha_{ij} z_{jl} (\gamma_{i,s,l} (1 - \gamma_{i,s,l}) + \gamma_{i,s-1,l} (1 - \gamma_{i,s-1,l})), & \text{if } s = 2, \dots, c_i. \end{cases} \\ \frac{\partial A_{3,i,s}(\mathbf{z}_{jl})}{\partial z_{kl}} &= \begin{cases} \alpha_{ik} z_{jl} \gamma_{i,1,l} (1 - \gamma_{i,1,l}), & \text{if } s = 1; \\ \alpha_{ik} z_{jl} (\gamma_{i,s,l} (1 - \gamma_{i,s,l}) + \gamma_{i,s-1,l} (1 - \gamma_{i,s-1,l})), & \text{if } s = 2, \dots, c_i \end{cases} \end{aligned}$$

On the other hand, the elements of the corresponding Hessian matrix are given by

$$\frac{\partial^2 A_{3,i,s}(\mathbf{z}_{jl})}{\partial z_{jl}^2} = \begin{cases} \alpha_{ij} \gamma_{i,1,l} (1 - \gamma_{i,1,l}) (2 - \alpha_{ij} z_{jl} (1 - 2\gamma_{i,1,l})), & \text{if } s = 1; \\ \alpha_{ij} [\gamma_{i,s,l} (1 - \gamma_{i,s,l}) (2 - \alpha_{ij} z_{jl} (1 - 2\gamma_{i,s,l})) + \\ \quad + \gamma_{i,s-1,l} (1 - \gamma_{i,s-1,l}) (2 - \alpha_{ij} z_{jl} (1 - 2\gamma_{i,s-1,l}))], & \text{if } s = 2, \dots, c_i. \end{cases}$$

$$\begin{aligned}
\frac{\partial^2 A_{3,i,s}(z_{jl})}{\partial z_{jl} \partial z_{kl}} &= \begin{cases} \alpha_{ik} \gamma_{i,1,l} (1 - \gamma_{i,1,l}) (1 - \alpha_{ik} z_{jl} (1 - 2\gamma_{i,1,l})), & \text{if } s = 1; \\ \alpha_{ik} [\gamma_{i,s,l} (1 - \gamma_{i,s,l}) (1 - \alpha_{ij} z_{jl} (1 - 2\gamma_{i,s,l})) + & \text{if } s = 2, \dots, ci. \\ + \gamma_{i,s-1,l} (1 - \gamma_{i,s-1,l}) (1 - \alpha_{ij} z_{jl} (1 - 2\gamma_{i,s-1,l}))], \end{cases} \\
\frac{\partial^2 A_{3,i,s}(z_{jl})}{\partial z_{kl}^2} &= \begin{cases} -\alpha_{ik}^2 z_{jl} \gamma_{i,1,l} (1 - 3\gamma_{i,1,l} + 2\gamma_{i,1,l}^2), & \text{if } s = 1; \\ -\alpha_{ik}^2 z_j (\gamma_{i,s-1,l} - 3\gamma_{i,s-1,l}^2 + 2\gamma_{i,s-1,l}^3 + \gamma_{i,s,l} - 3\gamma_{i,s,l}^2 + 2\gamma_{i,s,l}^3), & \text{if } s = 2, \dots, ci. \end{cases}
\end{aligned}$$